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DETERMINING PROPERTIES OF NONLINEAR MICROSTRUCTURED MATERIALS BY MEANS OF SOLITARY WAVES

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Abstract - We study an inverse problem to identify nonlinear microstructured materials involving nonlinearities in both macro- and microscales by means of information gathered from two solitary waves with different velocities. We state a uniqueness and stability theorem for the inverse problem.

1. INTRODUCTION

Microstructured materials like alloys, crystallites, ceramics, functionally graded materials, etc. have gained wide application. Determination of parameters of these materials is a problem of great practical importance. For this purpose wave processes going on in macro-level could be used. For instance, in [4] linear microstructured materials were identified by means of harmonic waves and Gaussian wave packets.

The microstructure brings along dispersive effects in the wave propagation. In case the dispersion is balanced by nonlinearity, solitary waves may emerge. Existence of solitary waves in microstructured solids has been proved both theoretically [3, 5, 7, 8] and experimentally [7, 9]. This paper studies an inverse problem to identify properties of nonlinear microstructured solids by means of information gathered from two independent solitary waves.

We use a mathematical model of the microstructure, which was derived according to the Mindlin ideas [6] by means of the hierarchical approach due to Engelbrecht and Pastrone [1]. Denoting by u the macrodisplacement, the basic 1D governing equation for longitudinal waves reads

$$u_{tt} = bu_{xx} + \frac{\mu}{2}\left(u_x^2\right)_x + \delta\left(\beta u_{tt} - \gamma u_{xx}\right)_{xx} - \delta^{3/2}\frac{\lambda}{2}\left(u_{xx}^2\right)_{xx} . \quad (1)$$

Here b, μ, β, γ and λ are coefficients related to the material properties and δ is a geometric parameter related to the scale of the microstructure (for details see [5]). The inequalities

$$0 < b < 1\,, \quad \delta,\ \beta,\ \gamma > 0 \quad (2)$$

are valid for the parameters b, δ, β and γ. Eq. (1) involves hierarchically two wave operators $u_{tt} - bu_{xx} - \frac{\mu}{2}\left(u_x^2\right)_x$ and $\delta\left(\beta u_{tt} - \gamma u_{xx} - \delta^{1/2}\frac{\lambda}{2}u_{xx}^2\right)_{xx}$ characteristic to the macro- and microstructure, respectively. The influence of the macro- and microstructure to the wave propagation depends on the size of the scale parameter δ [1, 5].

The related equation for the deformation $v = u_x$ reads

$$v_{tt} = bv_{xx} + \frac{\mu}{2}\left(v^2\right)_{xx} + \delta\left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} - \delta^{3/2}\frac{\lambda}{2}\left(v_x^2\right)_{xxx} . \quad (3)$$

Our aim is to identify 5 coefficients b, μ, β, γ and λ in eq. (3) by means of data gathered from solitary waves. The quantity δ is assumed to be known. Note that from the mathematical point of view we could set $\delta = 1$ redefining in a suitable way β, γ and λ in (3). But from the physical point of view it makes sense to preserve δ in our computations to show the scale-dependence.

2. PROBLEM FORMULATION

Travelling wave solutions of (3) have the form $v(x,t) = w(x - ct)$ where c is the velocity of the wave and $w = w(\xi)$ satisfies the equation

$$(c^2 - b)w'' - \frac{\mu}{2}(w^2)'' - \delta(\beta c^2 - \gamma)w^{IV} + \delta^{3/2}\frac{\lambda}{2}[(w')^2]''' = 0\,. \quad (4)$$

Since we are interested in solitary wave solutions vanishing at infinity, we are looking for w from the space

$$\mathcal{W} = \{w \,:\, w \not\equiv 0\,;\; w^{(i)} - \text{continuous in } \mathbb{R},\; i = 0,\ldots,4\,;\; \lim_{|\xi|\to\infty} w^{(i)}(\xi) = 0,\; i = 0,1,2\}.$$

The existence and properties of the solutions in $\mathcal{W}$ were proved in [8] and in case $\lambda = 0$ also in [5]. Let us formulate two results from [5].

Lemma 1. *If* (4) *has a solution in* $\mathcal{W}$ *then* $\mu \neq 0$, $\beta c^2 - \gamma \neq 0$, $c^2 - b \neq 0$ *and* $\frac{c^2-b}{\beta c^2-\gamma} > 0$. *Moreover, eq.* (4) *in* $\mathcal{W}$ *is equivalent to the following equation of the first order*

$$(w')^2 - \alpha(w')^3 \;=\; \kappa^2 w^2 \left(1 - \frac{w}{A}\right)\,, \tag{5}$$

where

$$\kappa \;=\; \sqrt{\frac{c^2-b}{\delta(\beta c^2-\gamma)}}\,, \quad A \;=\; \frac{3(c^2-b)}{\mu}\,, \quad \alpha \;=\; \frac{2\delta^{1/2}\lambda}{3(\beta c^2-\gamma)}\,. \tag{6}$$

Theorem 1. *Let* $\mu \neq 0$, $\beta c^2 - \gamma \neq 0$ *and* $c^2 - b \neq 0$. *Eq.* (4) *has a solution in* $\mathcal{W}$ *if and only if*

$$\left(\frac{\beta c^2-\gamma}{c^2-b}\right)^3 \;>\; \frac{4\lambda^2}{\mu^2} \qquad (\textit{equivalently,}\quad |A\alpha\kappa| < 1)\,. \tag{7}$$

In case (7) *holds, the set of all solutions in* $\mathcal{W}$ *has the form* $\{w_C(\xi) = w_0(\xi + C) \,:\, C \in \mathbb{R}\}$, *where* $w = w_0 \in \mathcal{W}$ *is an infinitely differentiable function in* $\mathbb{R}$, *which has the following properties:*

(a) $\ln|w(\xi)| \;\sim\; -\kappa|\xi|$ *as* $|\xi| \to \infty$;

(b) $A^{-1}w(\xi) \in (0,1)$ *if* $\xi \neq 0$ *and* $w(0) = A$, *i.e.,* A *is the amplitude;*

(c) $Aw'(\xi) > 0$ *if* $\xi < 0$, $Aw'(\xi) < 0$ *if* $\xi > 0$ *and* $w'(0) = 0$; w' *has exactly two relative extrema occurring at* $\xi = \xi^- < 0$ *and* $\xi = \xi^+ > 0$ *such that* $w(\xi^-) = w(\xi^+) = \frac{2A}{3}$;

(d) $|w'(\xi)| < \frac{2|A|\kappa}{3} < \frac{2}{3|\alpha|}$ *for* $\xi \in \mathbb{R}$;

(e) $|w(\xi)| > |w(-\xi)|$ *for any* $\xi > 0$ *in case* $\mu\lambda > 0$ $(A\alpha > 0)$,
$|w(\xi)| < |w(-\xi)|$ *for any* $\xi > 0$ *in case* $\mu\lambda < 0$ $(A\alpha < 0)$;

(f) *if* $\lambda = 0$ $(\alpha = 0)$ *then* $w(\xi) = A\cosh^{-2}\left(\frac{\kappa\xi}{2}\right)$.

Note that if the nonlinearity in the microscale is included, i.e. $\lambda \neq 0$ then the solitary wave is *asymmetric* (statement (e)). This is illustrated in Fig. 1 and 2.

We remark that a single solitary wave doesn't contain enough information to recover all five unknowns $b, \mu, \beta, \gamma, \lambda$. Indeed, eq. (5) depends upon three parameters A, κ, α. Thus, measuring the whole wave $w(\xi)$ we can recover maximally A, κ and α. But system (6) has infinitely many solutions $b, \mu, \beta, \gamma, \lambda$ for given A, κ, α and c^2. Consequently, it is necessary to measure at least two waves with different c^2-s.

Let we be given two waves $w[c_1]$ and $w[c_2]$ with the velocities c_1 and c_2 satisfying the inequality $c_1^2 \neq c_2^2$. Let A_1 and A_2 stand for the amplitudes of these waves. Then, by (6) we obtain the system $3b + A_j\mu = 3c_j^2$, $j = 1,2$ for unknowns b and μ. The assumption $c_1^2 \neq c_2^2$ implies $A_1 \neq A_2$, hence this system is regular. Consequently, the coefficients b and μ are uniquely determined by amplitudes of two waves. Unfortunately, the amplitudes which depend only on b, μ and δ cannot be used to recover the other unknowns β, γ, λ. Thus, we have to gather some additional information from solitary waves.

First, let us choose some numbers w_{11} and w_{12} which lie between 0 and A_1. We register the time when the first wave attains the level w_{11}, the extremum $w = A_1$, and the time when it drops below the level w_{12}. Knowing the velocity c_1 we can then compute the relative coordinates $\xi = \eta_{11} > 0$ and

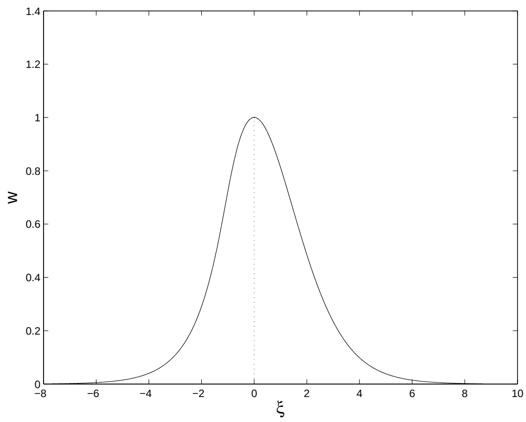

Figure 1:
Solitary wave in case $A = \kappa = 1$, $\alpha = 0.9$

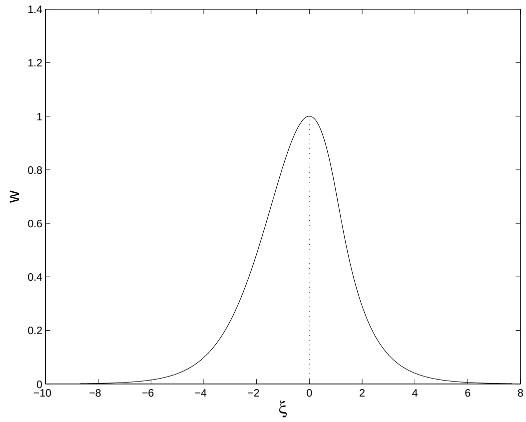

Figure 2:
Solitary wave in case $A = \kappa = 1$, $\alpha = -0.9$

$\xi = \eta_{12} < 0$ such that $w[c_1](\eta_{1l}) = w_{1l}, l = 1, 2$. Second, let us fix a number w_2 which lies between 0 and A_2. Similarly, for the second wave $w[c_2]$ we register the time when it either attains the level w_2 (case (1)) or drops below it (case (2)). Then, using the arrival time of the extremum $w = A_2$ and the velocity c_2 we can compute $\eta_2 \neq 0$ such that $w[c_2](\eta_2) = w_2$. Note that $\eta_2 > 0$ in case (1) and $\eta_2 < 0$ in case (2).

We pose the following *inverse problem*: given b, μ, the points (η_{1l}, w_{1l}), $l = 1, 2$ with $\eta_{11} > 0$, $\eta_{12} < 0$ on the graph of the first wave $w[c_1]$ and the point (η_2, w_2) with $\eta_2 \neq 0$ on the graph of the second wave $w[c_2]$, determine the triplet $S = (\beta, \gamma, \lambda)$.

3. PRELIMINARIES

Let us introduce some notation. We give to intervals of real numbers the following generalized meaning:

$$(d, e) = \{x \,:\, d < x < e\} \quad \text{in case } \; d < e\,; \quad (d, e) = \{x \,:\, e < x < d\} \qquad \text{in case } \; d > e.$$

As usual, $[d, e) = (d, e) \cup \{d\}, (d, e] = (d, e) \cup \{e\}$ and $[d, e] = (d, e) \cup \{d; e\}$.

In turns out that it is easier for our purposes to operate with the inverses of the solitary wave solutions w than w themself. Observing Thm. 1 we see that the function $w(\xi)$ has two inverses $\xi^+(w)$ and $\xi^-(w)$ which are defined for $w \in (0, A]$ and satisfy $\xi^\pm(A) = 0$, $\xi^+(w) > 0$, $\xi^-(w) < 0$ for $w \in (0, A)$ and $\lim\limits_{w \to 0^+} \xi^\pm(w) = \pm\infty$. Moreover, $A\xi^{+'}(w) < 0$, $A\xi^{-'}(w) > 0$ and $\xi^+(w)$, $\xi^-(w)$ have inflection points at $w = 2A/3$. We also emphasize that $\operatorname{sign} w = \operatorname{sign} A$ and $1 - \frac{w}{A} \in [0, 1)$.

By (5), the derivative of $\xi(w) = \xi^\pm(w)$ is a solution to the following equation for fixed $w \in (0, A)$:

$$\xi'(w) - \alpha \;=\; \kappa^2 w^2 \left(1 - \frac{w}{A}\right) [\xi'(w)]^3 \,. \tag{8}$$

It is possible to solve (8) for $\xi'(w)$ using Cardano formulas. But this leads to an expression which is too complex for integrating. Therefore we express $\xi(w)$ in a form of a functional series.

Lemma 2. *The formula*

$$\xi(w) \;=\; \frac{a_0}{\kappa} I_0(w) + \alpha \sum_{i=0}^{\infty} a_{i+1} (\alpha\kappa)^i I_{i+1}(w) \tag{9}$$

holds for $\xi = \xi^\pm$, *where* $I_i(w) = I_i[A](w)$ *are the following* w- *(and* A-*) dependent functions*

$$I_i(w) = \int_A^w \left[\tau \sqrt{1 - \frac{\tau}{A}}\right]^{i-1} d\tau = \begin{cases} -2 \ln \left[\sqrt{\frac{A}{w}} \left(1 + \sqrt{1 - \frac{w}{A}}\right)\right] & \text{if } i = 0 \\ 2A^i \sum\limits_{j=0}^{i-1} \binom{i-1}{j} (-1)^{j+1} \frac{\left(1 - \frac{w}{A}\right)^{\frac{i+1}{2} + j}}{i + 2j + 1} & \text{if } i \geq 1 \end{cases} \tag{10}$$

and a_i are generated by the recursive formulas

$$a_0^3 - a_0 = 0\,, \quad a_1 = (1-3a_0^2)^{-1}\,, \quad a_i = (1-3a_0^2)^{-1} \sum_{\substack{0\le i_1,i_2,i_3<i \\ i_1+i_2+i_3=i}} a_{i_1}a_{i_2}a_{i_3}\,, \; i\ge 2\,. \tag{11}$$

More precisely, the sequences a_i starting with $a_0 = 1$ and $a_0 = -1$ give the functions ξ^- and ξ^+, respectively. The series in (9) converges uniformly for any $w \in [0, A]$.

The main ideas of the proof of Lemma 2 are as follows. We introduce the new variables y and $\zeta = \zeta(y)$ by

$$y = \alpha\kappa w\sqrt{1-\frac{w}{A}}\,, \quad \xi'(w) = \frac{1}{\kappa w\sqrt{1-\frac{w}{A}}}\,\zeta\left(\alpha\kappa w\sqrt{1-\frac{w}{A}}\right). \tag{12}$$

Then eq. (8) for $\xi'(w)$ is equivalent to the cubic equation

$$[\zeta(y)]^3 - \zeta(y) + y \,=\, 0 \tag{13}$$

for $\zeta(y)$. We expand the solution of (13) in the form of the Taylor series $\zeta(y) = \sum\limits_{i=0}^{\infty} a_i y^i$. It can be shown that this series is uniformly convergent in every compact subset of $(-2/3, 2/3)$. Plugging it into (13) we obtain (11). Finally, substituting y and ζ by (12) in $\zeta(y) = \sum\limits_{i=0}^{\infty} a_i y^i$ and integrating we deduce (9).

The first addend in (9) $\xi_0(w) := \frac{a_0}{\kappa} I_0(w) = \frac{\pm I_0(w)}{\kappa}$ represents the two inverses of the symmetric bell-shaped solution, which occurs in case $\lambda = \alpha = 0$ (case (f) of Thm. 1). By the relations $I_0(w) \sim -\frac{2}{\sqrt{|A|}}\sqrt{|w-A|}$, $I_i(w) = o\,(I_{i-1}(w))$ as $w \to A$ and the uniform convergence of (9) we have

$$\xi(w) \sim -\frac{\sqrt{|w-A|}}{|A|}\left[\frac{2a_0}{\kappa} + \frac{A\alpha}{2}\sqrt{|w-A|}\right] \quad \text{as } \; w\to A. \tag{14}$$

In view of (6) we transform (8) to the form

$$3\delta(\beta c^2 - \gamma)\xi'(w) - 2\delta^{3/2}\lambda \,=\, \{3(c^2-b)w^2 - \mu w^3\}[\xi'(w)]^3\,. \tag{15}$$

To emphasize the dependence of $\xi^{\pm}(w)$ on the triplet $S = (\beta, \gamma, \lambda)$ and the velocity c we write $\xi^{\pm}(w) = \xi^{\pm}[S, c](w)$y.

Let us formulate a basic proposition providing a priori estimate for the difference of two triplets S in terms of differences of the variables w and ξ.

Proposition. *Let $c_1^2 \neq c_2^2$ and we be given two triplets $S^i = (\beta^i, \gamma^i, \lambda^i)$, $i = 1, 2$. Then for any $w_{11}^i, w_{12}^i \in (0, A_1) = \big(0\,,\, 3(c_1^2-b)/\mu\big)$ and $w_2^i \in (0, A_2) = \big(0\,,\, 3(c_2^2-b)/\mu\big)$, $i = 1, 2$ the following estimate holds:*

$$\max\Big\{\delta|\beta^1-\beta^2|\,;\,\delta|\gamma^1-\gamma^2|\,;\,\delta^{3/2}|\lambda^1-\lambda^2|\Big\} \le N(d^1,d^2)\varepsilon_\xi \,+\, \overline{N}(d^1,d^2)\varepsilon_w \tag{16}$$

where $d^i = (w_{11}^i, w_{12}^i, w_2^i, \xi_{11}^i, \xi_{12}^i, \xi_2^i)$, $i = 1, 2$ contain the components

$$\xi_{11}^i = \xi^+[S^i, c_1](w_{11}^i)\,, \;\; \xi_{12}^i = \xi^-[S^i, c_1](w_{12}^i)\,, \;\; \xi_2^i = \xi^{\pm}[S^i, c_2](w_2^i)\,, \; i = 1, 2\,. \tag{17}$$

Moreover,

$$\varepsilon_\xi = |\xi_{11}^1-\xi_{11}^2| + |\xi_{12}^1-\xi_{12}^2| + |\xi_2^1-\xi_2^2|\,, \; \varepsilon_w = |w_{11}^1-w_{11}^2| + |w_{12}^1-w_{12}^2| + |w_2^1-w_2^2|, \tag{18}$$

and the functions N, $\overline{N}$ are bounded in every compact subdomain of D^2 where $D = (0, A_1)^2 \times (0, A_2) \times (0, \infty) \times (0, -\infty) \times (0, \pm\infty)$.

Let us describe the ideas of proof of Proposition. We note that eq. (15) yields a 3×3 linear system for S if we take it with some arguments $w=\omega_{11},\omega_{12},\omega_2$ and the values $\xi'=[\xi^+[S,c_1](\omega_{11})]'$ $\xi'=[\xi^-[S,c_1](\omega_{12})]'$, $\xi'=[\xi^\pm[S,c_2](\omega_2)]'$, respectively. The determinant of this system is bounded away from zero owing to the different signs of the values $\xi'=[\xi^+[S,c_1](\omega_{11})]'$ and $\xi'=[\xi^-[S,c_1](\omega_{12})]'$. We can nicely go over from ξ' to ξ in this system making use of mean value theorems. As a result we can derive an estimate of S in terms of variables w and ξ. To derive an estimate (16) we use a similar technique writing down and analyzing a corresponding 3×3 system for the difference of S^1 and S^2.

4. MAIN RESULTS

The inverse problem posed in Sec. 2 can be written in the form of the system of nonlinear equations

$$\xi^+[S,c_1](w_{11})=\eta_{11}\,,\quad \xi^-[S,c_1](w_{12})=\eta_{12}\,,\quad \xi[S,c_2](w_2)=\eta_2 \tag{19}$$

for the triplet $S=(\beta,\gamma,\lambda)$ with the data $d=(w_{11},w_{12},w_2,\eta_{11},\eta_{12},\eta_2)$. Here $\eta_{11}>0$, $\eta_{12}<0$ and $\xi[S,c_2](w_2)=\xi^+[S,c_2](w_2)$ in case $\eta_2>0$ and $\xi[S,c_2](w_2)=\xi^-[S,c_2](w_2)$ in case $\eta_2<0$.

Important questions related to the inverse problem are the uniqueness and stability with respect to the errors of data. We take two types of errors into account. The first type is related to the inaccuracy of fixing the levels of measurement of the waves. Levels used in the computations differ somewhat from the values w_{11},w_{12},w_2, where the actual measurements are performed. Let us denote these approximate levels by $\widetilde{w}_{11},\widetilde{w}_{12},\widetilde{w}_2$. The second type is related to the inaccuracy of the measurement of time moments during the experiment. This leads to errors in η-s. Let us denote by $\widetilde{\eta}_{11},\widetilde{\eta}_{12},\widetilde{\eta}_2$ the approximate values of $\eta_{11},\eta_{12},\eta_2$ obtained by means of measurements. Summing up, instead of (19) we solve the problem

$$\xi^+[\widetilde{S},c_1](\widetilde{w}_{11})=\widetilde{\eta}_{11}\,,\quad \xi^-[\widetilde{S},c_1](\widetilde{w}_{12})=\widetilde{\eta}_{12}\,,\quad \xi[\widetilde{S},c_2](\widetilde{w}_2)=\widetilde{\eta}_2 \tag{20}$$

with the approximate data $\widetilde{d}=(\widetilde{w}_{11},\widetilde{w}_{12},\widetilde{w}_2,\widetilde{\eta}_{11},\widetilde{\eta}_{12},\widetilde{\eta}_2)$ and the solution $\widetilde{S}=(\widetilde{\beta},\widetilde{\gamma},\widetilde{\lambda})$. The solution is stable with respect to the data if $\widetilde{d}\to d$ implies $\widetilde{S}\to S$.

Proposition of Sec. 3 immediately implies the following

Theorem 2
(i) *The solution of the inverse problem is unique.*
(ii) *The solution is stable with respect to the data and satisfies the estimate*

$$\max\left\{\delta|\beta-\widetilde{\beta}|\,;\,\delta|\gamma-\widetilde{\gamma}|\,;\,\delta^{3/2}|\lambda-\widetilde{\lambda}|\right\}\le N(d,\widetilde{d})\varepsilon_\eta+\overline{N}(d,\widetilde{d})\varepsilon_w \tag{21}$$

where N, $\overline{N}$ are the functions from Proposition and

$$\varepsilon_\eta=|\eta_{11}-\widetilde{\eta}_{11}|+|\eta_{12}-\widetilde{\eta}_{12}|+|\eta_2-\widetilde{\eta}_2|\,,\ \varepsilon_w=|w_{11}-\widetilde{w}_{11}|+|w_{12}-\widetilde{w}_{12}|+|w_2-\widetilde{w}_2|. \tag{22}$$

Since the coefficients N and $\overline{N}$ are bounded in every compact subdomain of D^2, the solution of (19) is uniformly Lipschitz-continuous with respect to the data d in every compact subdomain of D.

The experiment related to the inverse problem involves measurement of the first wave at both sides of the extremum $w[c_1]=A_1$. This leads to functions $\xi[S,c_1]$ with different superscripts $+,-$ in the first two equations of (19). As we mentioned, the values of $\xi[S,c_1]$ with different signs enables us to estimate a determinant corresponding to the system from below in the proof of Proposition. We now ask, how is the situation when the first wave is measured twice from a single side of the extremum? In this case the system of equations corresponding to the inverse problem is

$$\xi[S,c_1](w_{11})=\eta_{11}\,,\quad \xi[S,c_1](w_{12})=\eta_{12}\,,\quad \xi[S,c_2](w_2)=\eta_2 \tag{23}$$

where the functions $\xi[S,c_1](w_{1l})$ occurring in first two equations have the common superscript: either $+$ or $-$, depending on $\operatorname{sign}\eta_{11}=\operatorname{sign}\eta_{12}$, and $w_{11}\ne w_{12}$. In case $w_{11},w_{12}\in[2A_1/3,A_1)$, using the strict monotonicity of $\xi[S,c_1]'(w)$ in the interval $(2A/3,A)$, it is again possible to estimate in a proper way the determinant from below and prove the uniqueness and stability for the system (23). However, in the general case of w_{11},w_{12} and w_2 the uniqueness for (23) doesn't hold. Let us provide a counter-example.

Counter-example. Due to (a) in Thm. 1 we have

$$\xi^{\pm}[S,c](w) \sim \mp\frac{1}{\kappa}\ln|w| \quad \text{as} \;\; w\to 0. \tag{24}$$

For given two triplets $S^i=(\beta^i,\gamma^i,\lambda^i)$, $i=1,2$ we denote

$$\kappa_j^i=\sqrt{\frac{c_j^2-b}{\delta(\beta^i c_j^2-\gamma^i)}}\,,\;\alpha_j^i=\frac{2\delta^{1/2}\lambda^i}{3(\beta^i c_j^2-\gamma^i)}\,,\;A_j=\frac{3(c_j^2-b)}{\mu}\,,\quad i,j=1,2. \tag{25}$$

We now observe that for any $\epsilon>0$ and $a_{01},a_{02}\in\{-1;1\}$ we can find S^i, $i=1,2$ such that

$$\begin{aligned}
&\beta^1\neq\beta^2,\,\gamma^1\neq\gamma^2,\,\lambda^1\neq\lambda^2\,,\quad |A_j\alpha_j^i\kappa_j^i|<1\,,\; i,j=1,2\,,\\
&\text{sign}\left[2a_{0j}\left(\tfrac{1}{\kappa_j^1}-\tfrac{1}{\kappa_j^2}\right)+\tfrac{A_j(\alpha_j^1-\alpha_j^2)}{2}\tfrac{\epsilon}{2}\right]\;=\;\text{sign}\left[a_{0j}\left(\tfrac{1}{\kappa_j^1}-\tfrac{1}{\kappa_j^2}\right)\right]\,,\; j=1,2.\\
&\text{sign}\left[2a_{0j}\left(\tfrac{1}{\kappa_j^1}-\tfrac{1}{\kappa_j^2}\right)+\tfrac{A_j(\alpha_j^1-\alpha_j^2)}{2}\,\epsilon\right]\;=\;-\text{sign}\left[a_{0j}\left(\tfrac{1}{\kappa_j^1}-\tfrac{1}{\kappa_j^2}\right)\right]\,,\; j=1,2.
\end{aligned} \tag{26}$$

Let us fix some pair $a_{01},a_{02}\in\{-1;1\}$. Choosing sufficiently small $\epsilon>0$ and triplets S^i, $i=1,2$ satisfying (26) from relations (14) and (24) we deduce the inequalities

$$\begin{aligned}
\text{sign}\left[\xi[S^1,c_j](\omega_{1j})-\xi[S^2,c_j](\omega_{1j})\right] &= -\text{sign}\left[\xi[S^1,c_j](\omega_{2j})-\xi[S^2,c_j](\omega_{2j})\right]\\
&= \text{sign}\left[\xi[S^1,c_j](\omega_{3j})-\xi[S^2,c_j](\omega_{3j})\right]\,,\; j=1,2
\end{aligned} \tag{27}$$

where $|\omega_{1j}|=\epsilon$, $|\omega_{2j}-A_j|=\epsilon$, $|\omega_{3j}-A_j|=\epsilon/2$, $j=1,2$ and $\xi[S^i,c_j]=\xi^+[S^i,c_j]$ in case $a_{0j}=-1$ and $\xi[S^i,c_j]=\xi^-[S^i,c_j]$ in case $a_{0j}=1$. Relations (27) imply that there exist $w_{11}\in(\omega_{11},\omega_{12})$, $w_{12}\in(\omega_{12},\omega_{13})$ and $w_2\in(0,A_2)$ such that

$$\xi[S^1,c_1](w_{1l})-\xi[S^2,c_1](w_{1l})=\xi[S^1,c_2](w_2)-\xi[S^2,c_2](w_2)=0\,,\; l=1,2. \tag{28}$$

Consequently, (23) has two solutions $S^1\neq S^2$ for such $w_{11}\neq w_{12}$ and w_2.

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REFERENCES

1. J. Engelbrecht and F. Pastrone, Waves in microstructured solids with strong nonlinearities in microscale. *Proc. Estonian Acad. Sci. Phys. Math.* (2003) **52**, 12–20.
2. J. Engelbrecht, F. Pastrone and A. Berezovski, Wave propagation in microstructured solids. *Proc. WCU2003 (Paris)* http://www.afa.asso.fr/wcu2003/procs/website/index.html, pp. 495–497.
3. V. I. Erofeev, *Wave Processes in Solids with Microstructure*, World Scientific, Singapore, 2003.
4. J. Janno and J. Engelbrecht, Waves in microstructured solids: Inverse problems, *Institute of Cybernetics at Tallinn UT. Research Report Mech* 250/2003, 12p. (submitted)
5. J. Janno and J. Engelbrecht, Solitary waves in nonlinear microstructured materials, *Institute of Cybernetics at Tallinn UT. Research Report Mech* 259/2004, 10p. (submitted)
6. R. D. Mindlin, Micro-structure in linear elasticity. *Arch. Rat. Mech. Anal.* (1964) **16** , 51–78.
7. A. V. Porubov, *Amplification of Nonlinear Strain Waves in Solids*, World Scientific, Singapore, 2003.
8. A. V. Porubov and F. Pastrone, Non-linear bell-shaped and kink-shaped strain waves in microstructured solids. *Int. J. Non-Linear Mech.* (2004) **39**, 1289–1299.
9. A. M. Samsonov, G. V. Dreiden and I. V. Semenova, On the existence of bulk solitary waves in plexiglas. *Proc. Estonian Acad. Sci. Phys. Math.* (2003) **52**, 115–124.