Homework due Wednesday
Test case on web page

Today: Interpolation Functions/Elements

Geometry:
1-D

2-D

3-D

Whole idea: represent field variable over a small region

Fig 5.5a)
Will have one term in interpolation function for each node.

Virtually all F.E. analysis uses polynomial basis functions.

1-D 2 nodes: \( u(x) = a_0 + a_1 x \)  Linear
3 nodes: \( u(x) = a_0 + a_1 x + a_2 x^2 \)  Quadratic

etc

2-D 3 nodes:
2 nodes: \( u(x,y) = a_0 + a_1 x + a_2 y \)  Linear
6 nodes: \( u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 \)

Can choose polynomial you want, but usually want "complete" polynomial like the two above. If not complete, want them to be symmetric.

Fig 5-8 "Tree" of complete polynomials.

Bear in mind: idea is to use as simple function as we can.

Q: 4 nodes in 2-D problem? What kind of polynomial to use?
A: Bilinear.
Deriving Interpolation functions
At least 3 methods:

1) Collocation procedure (pg 149)
2) Lagrange Polynomials
3) Natural coordinate systems

1) Collocation procedure - match at specified points:

\[ u_1^e \quad u_2^e \quad u_3^e \]

\[ 0 \quad 3 \quad 2 \]

\[ x = 0 \quad x = \frac{1}{2} \quad x = \ell \]

a) Assume polynomial

\[ u^e(x) = a_0 + a_1 x + a_2 x^2 \]

b) "Match" \( u_1^e, u_2^e, u_3^e \) at \( x = 0, \frac{1}{2}, \ell \):

\[ u_1^e = u^e(x = 0) = a_0 \]
\[ u_2^e = u^e(x = \frac{1}{2}) = a_0 + \frac{1}{2} a_1 + \frac{1}{2} a_2 \frac{1}{4} \]
\[ u_3^e = u^e(x = \ell) = a_0 + \ell a_1 + \ell^2 a_2 \]

C) Solve for \( a_0, a_1, a_2 \):

\[
\begin{bmatrix}
    u_1^e \\
    u_2^e \\
    u_3^e
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & 0 \\
    1 & \frac{1}{2} & \frac{1}{4} \\
    1 & \ell & \ell^2 / 4
\end{bmatrix}
\begin{bmatrix}
    a_0 \\
    a_1 \\
    a_2
\end{bmatrix}
\]
1) Write the polynomial in form

\[ u^e(x) = \sum_i L_i^e(x) \{ \begin{array}{c} u_i^e \\ u_2^e \\ u_3^e \end{array} \} \]

\[ = \begin{bmatrix} N_1(x) \\ N_2(x) \\ N_3(x) \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{bmatrix} \]

2) Lagrange Polynomials

Can write interpolating function as

\[ L_i^e(x) = \prod_{\substack{j=1 \atop j \neq i}}^n \frac{(x-x_j)}{(x_i-x_j)} \quad \text{(Eqn 5.53)} \]

Then:

\[ u^e(x) = \sum_i L_i^e(x) u_i^e \]

Example: quadratic rod element

\[ L_i^e(x) = \begin{cases} \frac{x-x_i}{l} & \text{if } x \leq x_i \\ \frac{x-x_i}{l} & \text{if } x \geq x_i \end{cases} \]

\[ L_i(x) = \frac{(x-x_i)}{(0-x_i)} \begin{bmatrix} \frac{l}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{2}{l} \frac{(x-x_i)}{(0-x_i)} \]

\[ L_i(x) = \frac{(x-x_i)}{(0-x_i)} \begin{bmatrix} \frac{l}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{2}{l} \frac{(x-x_i)}{(0-x_i)} \]
\[ L_2(x) = \left( \frac{x - 0}{x - e} \right) \left( \frac{x - le/2}{x - le} \right) \]
\[ = \frac{2}{e^2} (x) \left( x - le/2 \right) \]
\[ L_3(x) = \left( \frac{x - 0}{x - le/2} \right) \left( \frac{x - le}{x - le} \right) \]
\[ = \frac{4}{e^2} (x) \left( x - le \right) = \frac{4}{e^2} (x)(le-x) \]

Notice: \[ L_i(x) \] has property that
\[ L_i(x) = \begin{cases} \frac{1}{i} & x = x_i \text{ own node} \\ 0 & x = x_j \text{ any other node} \end{cases} \]

is general property of polynomial interpolation functions.