Homework - due Fri 10/28

Put isoparametric linear quad elements into program to solve one heat conduction
  - to add type 3 B.C.
  - Heat generation

\[ \int_0^1 \{ \mathbf{N} \} \mathbf{q}' \, d\Omega \]

Today: Start w/ Fluid Flow

Must satisfy

Steady, incompressible fluid

- Continuity (mass conservation)
  \[ \mathbf{\nabla} \cdot \mathbf{u} = \beta \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \]
- Momentum conservation

For low velocities, advection is negligible and the equation is linear:

\[ \frac{\partial}{\partial x} (\sigma_x - P) + \frac{\partial \tau_{xy}}{\partial y} = 0 \]
\[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial}{\partial y} (\sigma_y - P) = 0 \]
P - thermodynamic pressure
Constitutive relations for fluid relate velocities to stresses

For a Newtonian fluid:

\[ \sigma_x = 2 \eta \frac{\partial u}{\partial x} \]
\[ \sigma_y = 2 \eta \frac{\partial v}{\partial y} \]
\[ \tau_{xy} = \eta \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]

\( \eta \) = dynamic viscosity - property of fluid

Boundary conditions:
Type 1 - specified velocity
\[ u = g(x, y) \quad v = h(x, y) \] on \( S \)

Type 2 - specified traction
\[ \bar{\sigma}_x = (\sigma_x - p) \hat{n}_x + \tau_{xy} \hat{n}_y \] on \( S_2 \)
\[ \bar{\sigma}_y = (\sigma_y - p) \hat{n}_y + \tau_{xy} \hat{n}_x \]
\[ \sigma_x - p \]
\[ \sigma_y - p \]
These eqns are in terms of primary variables \( u, v, p \)
(An alternative is to transform these eqns into a stream function formulation (see pg 42).)

3 eqns, 3 unknowns

Solve using Galerkin M.W.R.
F.E.

On each element, let

\[
\begin{align*}
U^e &= LN^e \{ U \} \\
V^e &= LN^e \{ V \} \\
\mathbf{P}^e &= LN^p \{ \mathbf{P} \}
\end{align*}
\]

\( N \) and \( N^p \) are different shape functions

Later we'll see that the order of \( LN^e \) must be at least one degree higher than \( N^p \).

Multiply continuity eqn by \( N^p \), momentum eqn by \( N \) and integrate over element, set = 0.
\[
\int \left[ \frac{\partial}{\partial x} (\sigma_x - p) + \frac{\partial \tau_{xy}}{\partial y} \right] N_i \, d\Omega = 0
\]
\[
\int \left[ \frac{\partial \tau_{xy}}{\partial x} + 2 \left( \sigma_y - p \right) \right] N_i \, d\Omega = 0
\]
\[
\int \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) N_i^p \, d\Omega = 0
\]
\[
\frac{\partial}{\partial x} \left( N_i \right) \cdot \mathbf{v} \, d\Omega = 0
\]
\[
\int \left\{ N_i \right\} \cdot \mathbf{v} \, d\Omega = 0
\]
\[
\int \left\{ N_i \right\} \cdot \mathbf{v} \, d\Omega = 0
\]

Use integration by parts (Green's theorem)

\[
\mathbf{v} \cdot \int \left\{ N_i \right\} \mathbf{\hat{x}} \, d\Gamma - \int \left\{ \nabla N_i \right\} \cdot \mathbf{\hat{x}} \, d\Omega = 0
\]

Introduce constitutive relations

\[
\int \left[ (2\nu \sigma_{xy} - p) \frac{\partial N_i}{\partial x} + (\frac{2
u}{\gamma} + \frac{2v}{\alpha}) \frac{\partial^2 N_i}{\partial y^2} \right] d\Omega = 0
\]

\[
\int \sigma_x N_i \, d\Gamma
\]

Similar for y-momentum - see 9.23c.6
Introduce interpolation functions $K_{2n}$

$$
\int_{\Omega} \left[ \int_{\partial \Omega} \left( \frac{\partial u}{\partial n} + \frac{\partial v}{\partial n} \right) d\Gamma \right] d\Omega + \int_{\Omega} \int_{\partial \Omega} \left( \frac{\partial^2 u}{\partial n^2} + \frac{\partial^2 v}{\partial n^2} \right) d\Omega d\Gamma
$$

$x$-momentum

$$
\int_{\Omega} \left\{ \frac{\partial u}{\partial x} \right\}_{x \in \Omega} \int_{\partial \Omega} \left\{ \frac{\partial^2 u}{\partial x^2} \right\} d\Gamma = 0
$$

$$
\int_{\Omega} \left\{ \frac{\partial u}{\partial x} \right\}_{x \in \Omega} \int_{\partial \Omega} \left\{ \frac{\partial^2 u}{\partial x^2} \right\} d\Gamma = \int_{S_2} \tilde{\sigma}_x N_0 d\Gamma
$$

The $y$-momentum eqn is

$$
[K_{12}]^T \{ u \} + [K_{11} + 2K_{22}] \{ v \} + \left[ L_2 \right] \{ P \} = \int_{S_2} \tilde{\sigma}_y N_0 d\Gamma
$$

$$
L_2 = \int_{\Omega} \left\{ \frac{\partial N}{\partial y} \right\}_{y \in \Omega} \int_{\partial \Omega} \left\{ \frac{\partial^2 \text{NP}}{\partial y^2} \right\} d\Gamma
$$

Continuity eqn (recall weight vector is \{ N_0 \})

$$
[L_1]^T \{ u \} + [L_2]^T \{ v \} = \{ 0 \}
$$

Where the system can be put in matrix form as

\[ B \]
eqn 9.25
pg 439

\[
\begin{bmatrix}
    r & r & s \\
    [2K_{u}+K_{z}] & [K_{z}] & [L_{1}] \\
    -[K_{z}]^T & [K_{u}+2K_{z}] & [L_{2}] \\
    [L_{1}]^T & [L_{2}]^T & [O]
\end{bmatrix}
\]

For practical reasons \( R > S \)

Note \( K_{u}, K_{z}, K_{z} \) are \( r \times r \)

\( L_{1}, L_{2} \) are \( r \times s \)

If \( N \) are \( b \) bi-linear (Severndipity elements)
then \( r = 4 \)

If \( N^{p} \) is constant on element
then \( S = 1 \)

pg 440 \( \rightarrow \) definitions of \( K_{u}, K_{z}, K_{z} \)

\( L_{1}, L_{2} \)