Text: Many inverse problems can be formulated as an operator equation of the first kind

(1) \( \mathbf{Ay} = \mathbf{x} \)

where bounded linear operator \( \mathbf{A} : \mathbb{H} \rightarrow \mathbb{H} \) maps a Hilbert space \( \mathbb{H} \) to itself, and there is no continuous mapping of data \( \mathbf{x} \) into solution \( \mathbf{y} \). In practical calculations the possibility of stable recovery of \( \mathbf{y} \) from \( \mathbf{x} \) usually is justified by restricting the set of solutions in terms of boundedness, smoothness, monotonicity, etc. It is assumed that \( \mathbf{y} \) belongs to some subset \( \mathbb{M} \) in \( \mathbb{H} \). If there is a way to calculate any \( \mathbf{y} \) from \( \mathbb{M} \) using the approximate data \( \mathbf{x} \) from \( \mathbb{H} \) with an arbitrary small accuracy defined only by the magnitude of data error \( ||\mathbf{Ay} - \mathbf{x}|| \), then the problem (1) is uniformly regularized, and \( \mathbb{M} \) is called a uniform regularization set (URS). The choice of URS is of critical importance, since imposing an overly tight restriction can make the mathematical model meaningless, while choosing a too wide URS may result in an unacceptable instability. However, the necessary and sufficient conditions on whether \( \mathbb{M} \) is an URS are widely known only when the operator \( \mathbf{A} \) is completely continuous. In general case, regularization theory suggests several popular URS types, such as finite dimensional sets, compact sets, ellipsoids, band-limited sets for certain operators in \( L^2 \) space, and others, which all represent only sufficient conditions on URS for problems with a non-compact operator.

The main result of this paper provides necessary and sufficient conditions on convex URS for an arbitrary bounded operator \( \mathbf{A} \) in Hilbert space. The same conditions are proved to be sufficient for a non-convex set to be an URS for problem (1). Also presented an application of these results to ill-posed problems for partial differential equations and integral equations of the convolution type in \( L^2 \) space. It is shown that this theorem gives an easy way to form an URS by imposing restrictions on the balance between the spectral components of feasible solutions. Also are discussed several URS examples obtained from different decompositions of \( L^2 \) space, and numerical solution algorithms based on uniform regularization with an URS of arbitrary type.