Laplace Transforms

- The Laplace Transform (LT) is used to convert ______________________ into ____________________________ in independent variable t independent variable s
- The Laplace Transform (LT) is also used to convert the ___________________________ into the ___________________________

Transforms of Functions

Function of Time  Laplace Transform

<table>
<thead>
<tr>
<th>Function</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dirac delta $\delta(t)$</td>
<td>$\Leftrightarrow \frac{1}{s}$</td>
</tr>
<tr>
<td>Unit step $U(t)$</td>
<td>$\Leftrightarrow \frac{A}{s}$</td>
</tr>
<tr>
<td>Constant $A$</td>
<td>$\Leftrightarrow \frac{A}{s}$</td>
</tr>
</tbody>
</table>

Transforms of Functions

Function of Time  Laplace Transform

<table>
<thead>
<tr>
<th>Function</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit ramp $t$</td>
<td>$\Leftrightarrow \frac{1}{s^2}$</td>
</tr>
<tr>
<td>Unit parabola $t^2$</td>
<td>$\Leftrightarrow \frac{1}{s^3}$</td>
</tr>
<tr>
<td>Higher powers of $t$</td>
<td>$\Leftrightarrow \frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>Exponential $e^{-at}$</td>
<td>$\Leftrightarrow \frac{1}{s + a}$</td>
</tr>
<tr>
<td>$e^{+at}$</td>
<td>$\Leftrightarrow \frac{1}{s - a}$</td>
</tr>
<tr>
<td>$te^{-at}$</td>
<td>$\Leftrightarrow \frac{1}{(s + a)^2}$</td>
</tr>
</tbody>
</table>
Transforms of Functions

Function of Time  Laplace Transform

$\sin \omega t \quad \Leftrightarrow \quad \frac{\omega}{s^2 + \omega^2}$

$\cos \omega t \quad \Leftrightarrow \quad \frac{s}{s^2 + \omega^2}$

LT Properties #1-#2

1) Multiplication by a constant
   \[ \mathcal{L}[k \, f(t)] = k \, F(s) \]

2) Summation
   \[ \mathcal{L}[f(t) + g(t)] = F(s) + G(s) \]

Product - note carefully!

\[ \mathcal{L}[f(t) \cdot g(t)] \neq F(s) \cdot G(s) \]

LT Properties #3-#4

3) Differentiation
   \[ \mathcal{L}\left[\frac{d f(t)}{d t}\right] = \left(\frac{1}{s}\right)F(s) - f(0) \]
   
   (\( f(0) \) is the initial condition)

4) Integration
   \[ \mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \left(\frac{1}{s}\right)F(s) \]
Solving Differential Eqns w/LT

\[ \dot{x} = ax + bu(t) \quad x(0) = x_0 \]

Take LT of both sides of equation,

\[ sX(s) - aX(s) = x_0 + bU(s) \]

Classic Solution to Differential Eqn

\[ \dot{x} = ax + bu(t) \quad x(0) = x_0 \]

Time-domain solution is (p.51):

- zero-input response
- zero-state response

Laplace Solution to SV Eqns

\[ \begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*} \]

Linear, time invariant state & output equations

Taking Laplace Transforms for both sides,
Laplace Solution to SV Eqns

Pre-multiply both sides of the equation by the inverse,

Taking inverse Laplace Transforms for each term
gives time domain solution.

Similarly, the time-domain solution is (p.62):

Matrix Exponential Properties

1. Solves equation
2. Equals identity matrix when $t=0$,
3. Is invertible (nonsingular) for all $t$ with inverse,
4. Is commutable with $A$,

Matrix Exponential Computation

1. Solve by Laplace Transforms:
2. Solve by power series (see text p.53)
3. Solve by Cayley-Hamilton theorem, i.e., every square matrix satisfies its own characteristic equation
4. Use Maple (or maybe Matlab??)

Solve by Laplace Transform

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{\text{adj}(sI - A)}{|sI - A|} = \frac{\text{cof}(sI - A)^T}{|sI - A|} = \begin{bmatrix} s + 3 & 2 \\ s + 1 & s + 1 \end{bmatrix}$$
Solve by Laplace Transform

\[ \mathcal{L}^{-1} \left( [sI - A]^{-1} \right) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \]

Check Cayley-Hamilton Theorem

\[ A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} \]

\[ \lambda^2 + 4 \lambda + 3 = 0 \rightarrow A^2 + 4A + 3I = 0 \]

Using Cayley Hamilton to Find Matrix Exponential

If \( A \) is of order \( n \), then for any function \( f(\lambda) \) we can construct a polynomial of degree \( n-1 \), such that \( g(\lambda) = f(\lambda) \) on the spectrum of \( A \), therefore any polynomial of \( A \) can be expressed as
Using Cayley Hamilton to Find Matrix Exponential - #1

\[ e^{At} = f(A) = g(A) = \alpha_0 I + \alpha_1 A \]

Using Cayley Hamilton to Find Matrix Exponential - #2

\[ |\lambda I - A| = \lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) \rightarrow \lambda = f(-1) = g(-1) = \alpha_0 + \alpha_1(-1) \]
\[ f(-3) = g(-3) = \alpha_0 + \alpha_1(-3) \]

\[
\begin{bmatrix}
0 & -1 \\
1 & -3
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\
\alpha_1
\end{bmatrix}
=
\begin{bmatrix}
e^{-t} \\
e^{-3t}
\end{bmatrix}
\]

Using Cayley Hamilton to Find Matrix Exponential - #3

\[
\begin{bmatrix}
\alpha_0 \\
\alpha_1
\end{bmatrix}
=
\begin{bmatrix}
1.5 & -0.5 \\
0.5 & -0.5
\end{bmatrix}
\begin{bmatrix}
e^{-t} \\
e^{-3t}
\end{bmatrix}
\]

\[ e^{At} = \begin{bmatrix}
e^{-t} & e^{-t} - e^{-3t} \\
0 & e^{-3t}
\end{bmatrix} \]

Find Matrix Exponential with Maple

```maple
> with(linalg):
Warning: new definition for norm
Warning: new definition for trace
> A := array([ [-1,2], [0,-3] ]);
> eAt := exponential(A,t);
A :=
\begin{bmatrix}
-1 & 2 \\
0 & -3
\end{bmatrix}
\]
\[ e^{At} := \begin{bmatrix}
e^{(-t)} & e^{(-t)} - e^{(-3t)} \\
0 & e^{(-3t)}
\end{bmatrix} \]
```