Stability - 2

- Jordan Form for Repeated Eigenvalues
- Energy-based Methods
- Lyapunov Functions

LTI Form of State Equations

\[
\begin{align*}
\dot{x}(t) &= A\dot{x}(t) + Bu(t) \\
y(t) &= C\dot{x}(t) + Du(t) \\
x(0) &= x_0
\end{align*}
\]

This system is **stable** if there exists

This system is **globally asymptotically stable** if

Zero-Input Solution

Recall that the matrix exponential and the initial condition determines the zero-state response,

\[
x(t) = e^{\Delta t} x_0 + \int_0^t e^{\Delta (t-\tau)} Bu(\tau) d\tau
\]

Each element of the matrix exponential \( e^{\Delta t} \) must

Diagonal Canonical Form for Distinct Eigenvalues

We have already seen that any matrix \( A \) can be transformed to the diagonal canonical form by a transformation matrix made up of eigenvectors,
Jordan Blocks for Repeated Eigenvalues

If a system has repeated eigenvalues, the strictly diagonal form $\hat{A}$ cannot always be obtained. Depending on the characteristics of the $A$ matrix, one or more Jordan blocks may be the closest to a diagonal form that can be obtained.

\[
J_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad J_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

Jordan Block Example #1a

\[
A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -1 & -1 \\ -1 & 2 & -4 \end{bmatrix}
\]

\[
>> [n,e]=eig(A)
n = \begin{bmatrix} 0.0000 & 0.0000 & -0.7071 \\ 0.7071 & 0.7071 & -0.0000 \\ 0.7071 & 0.7071 & 0.7071 \end{bmatrix}
e = \begin{bmatrix} ____ & 0 & 0 \\ 0 & ____ & 0 \\ 0 & 0 & ____ \end{bmatrix}
\]

Jordan Block Example #1b

\[
J = \text{inv}(T) \cdot A \cdot T
\]

\[
>> \text{inv}(T) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\]

Jordan Block Example #1c

\[
J = T^{-1} AT \rightarrow e^{Jt} = Te^{Jt}T^{-1}
\]

\[
>> J=\text{inv}(T) \cdot A \cdot T = \]

\[
\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}
\]
Jordan Block Example #1d

\[
[sI - J]^{-1} = \begin{bmatrix}
\_ & \_ & 0 \\
0 & \_ & 0 \\
0 & 0 & \_
\end{bmatrix}
\]

\[
L^{-1}([sI - J]^{-1}) = e^{st} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Jordan Block Example #2

\[
A = \begin{bmatrix}
-2 & -1 & 1 \\
0 & -2 & 0 \\
0 & 1 & -3
\end{bmatrix}
\]

\[
>> \quad [n,e]=eig(A)
\]

\[
n = \begin{bmatrix}
1.0000 & 0 & -0.7071 \\
0 & 0.7071 & 0 \\
0 & 0.7071 & 0.7071
\end{bmatrix}
\]

\[
e = \begin{bmatrix}
\_ & \_ & 0 \\
0 & \_ & 0 \\
0 & 0 & \_
\end{bmatrix}
\]

Theorem 6.3

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

Linear, time invariant state & output equations

System is stable iff

System is globally asymptotically stable iff

Energy-Based Methods

- See discussion on pp. 202-210 of text
- Energy-based methods have an intuitive relationship with the determination of stability, but ____________________________ of other methods
- Are used fairly extensively __________
**Energy-Based Example #1a**

The derivative of energy with respect to time is

\[ \dot{E} = \frac{1}{2} K x \dot{x} + \frac{1}{2} M v \dot{v} = K \dot{x}_1 \dot{x}_1 + M \dot{x}_2 \dot{x}_2 \]

For any real damper, force opposes motion, so \( B \geq 0 \)

**Energy-Based Example #1b**

If \( B > 0 \) we know that the system will eventually return to the equilibrium position.

If \( B = 0 \) we have \( \dot{E} = 0 \) and

**Energy-Based Example #1c**

If \( B > 0 \) we know that the system will eventually return to the equilibrium position.

If \( B = 0 \) we have \( \dot{E} = 0 \) and

**Lyapunov Methods**

- Lyapunov's 2nd (Direct) Method
  - uses a generalized energy-based approach
  - can be applied to________________________
    where energy is undefined

- Define a___________ Lyapunov Function of the states, \( V(\bar{x}) \)
Positive Definite and Semi-definite

- Positive definite $V(x)$
  - $V(x)$ is continuous with continuous partial derivatives and
- Positive semi-definite $V(x)$
  - $V(x)$ is continuous with continuous partial derivatives and

Negative Definite and Semi-definite

- Negative definite $V(x)$
  - $V(x)$ is continuous with continuous partial derivatives and
- Negative semi-definite $V(x)$
  - $V(x)$ is continuous with continuous partial derivatives and

Lyapunov’s 2nd (Direct) Method

If positive definite function $V(x)$ can be found such that $\dot{V}(x)$ is

If positive definite function $V(x)$ can be found such that $\dot{V}(x)$ is

Note carefully the wording – if you can find any function $V(x)$ that satisfies the conditions,

If you cannot find a $V(x)$ that satisfies the conditions, then