Stability

In order for a system to be stable (steady state response to a finite input is finite)

All poles of the transfer function must have negative real parts

Recall that poles of the transfer function are roots of the characteristic equation (denominator of transfer function)

Negative real roots

\[ae^{-r_1t} \quad \text{and} \quad be^{-r_2t}\]

What are the time domain responses due to these roots?
Zero real root

What are the time domain responses due to these roots?

\[ a \quad \text{and} \quad b e^{-r_2 t} \]

What if the finite input is a constant or step?

\[ a, \, b e^{-r_2 t}, \, \text{and} \, ct \]

Complex roots - Negative real part

\[ a e^{-\xi_0 t} \sin \left( \frac{1}{2} \omega_n \sqrt{1-\xi^2} t \right) \quad \text{and} \quad b e^{-\xi_0 t} \cos \left( \frac{1}{2} \omega_n \sqrt{1-\xi^2} t \right) \]

What are the time domain responses due to these roots?
Complex roots - Zero real part

What are the time domain responses due to these roots?

\[ a \sin(\omega_n t) \quad \text{and} \quad b \cos(\omega_n t) \]

What if the finite input is a sine wave at frequency \( \omega_n \)?

\[ a \sin(\omega_n t), b \cos(\omega_n t), \text{ and } \]
\[ c \sin(\omega_n t), d \cos(\omega_n t), \]

Complex roots - Positive real part

\[ a e^{\zeta \omega_n t} \sin \left( \tau \omega_n \sqrt{1 - \zeta^2} \right) \quad \text{and} \quad \]
\[ b e^{\zeta \omega_n t} \cos \left( \tau \omega_n \sqrt{1 - \zeta^2} \right) \]

What are the time domain responses due to these roots?

\[ - j \omega_n \sqrt{1 - \zeta^2} \quad \text{and} \quad + j \omega_n \sqrt{1 - \zeta^2} \]
Effect of Control on Stability

Is there an upper limit on $K_p$ for all roots of the characteristic equation of the overall transfer function to have negative real parts?

$$X_d(s) + \frac{1}{s(s^2 + 6s + 8)} X(s)$$

Roots of Characteristic Equation

$$s(s^2 + 6s + 8) + K_p = s^3 + 6s^2 + 8s + K_p = 0$$

<table>
<thead>
<tr>
<th>$K_p$</th>
<th>real root</th>
<th>complex roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4.4567</td>
<td>-0.7717 +/- j 0.7256</td>
</tr>
<tr>
<td>10</td>
<td>4.7608</td>
<td>-0.6196 +/- j 1.3102</td>
</tr>
<tr>
<td>20</td>
<td>5.2572</td>
<td>-0.3974 +/- j 1.9198</td>
</tr>
<tr>
<td>30</td>
<td>5.534</td>
<td>-0.233 +/- j 2.3166</td>
</tr>
<tr>
<td>35</td>
<td>5.6768</td>
<td>-0.1624 +/- j 2.4778</td>
</tr>
<tr>
<td>40</td>
<td>5.8083</td>
<td>-0.0958 +/- j 2.6225</td>
</tr>
<tr>
<td>45</td>
<td>5.9305</td>
<td>-0.0347 +/- j 2.7544</td>
</tr>
<tr>
<td>50</td>
<td>6.0449</td>
<td>0.0225 +/- j 2.8759</td>
</tr>
</tbody>
</table>

Stable for these values of $K_p$

Unstable for $K_p = 50$ (or more)

Use Matlab's `roots` function with different values of $K_p$. What is the largest $K_p$ for all roots to have negative real parts?
Roots of Polynomials

An \( n \)th order polynomial has at least one non-negative real root if:

- any coefficient of a lower power of \( s \) (\( s^{n-1}, s^{n-2}, \text{etc.} \)) is zero
- any coefficient has a different sign than the other coefficients

Stability - Routh-Hurwitz

\[ s \left( s^2 + 6s + 8 \right) + 50 = s^3 + 6s^2 + 8s + 50 = 0 \]

Alternatively, the Routh-Hurwitz criteria can be used to determine the limits on stability,

<table>
<thead>
<tr>
<th>( S^3 )</th>
<th>+1</th>
<th>+6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S^2 )</td>
<td>+8</td>
<td>+50</td>
</tr>
<tr>
<td>( S )</td>
<td>+6*8-50</td>
<td>+0</td>
</tr>
<tr>
<td>( S^0 )</td>
<td>-2/8</td>
<td>+50</td>
</tr>
</tbody>
</table>

Sign changes indicate roots with non-negative real parts
**Routh-Hurwitz Matrix**

Does \( s^4 + 2s^3 + 6s^2 + 8s + 50 = 0 \) have roots with negative real parts?

<table>
<thead>
<tr>
<th>( s^4 )</th>
<th>1</th>
<th>6</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^3 )</td>
<td>2</td>
<td>8</td>
<td>(implicit) 0</td>
</tr>
<tr>
<td>( s^2 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s^1 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s^0 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Odd powers of \( s \)

Every power of \( s \)

---

**Compute \( s^2 \) row**

Always use coefficients from 1st column

<table>
<thead>
<tr>
<th>( s^4 )</th>
<th>1</th>
<th>6</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^3 )</td>
<td>2</td>
<td>8</td>
<td>(implicit) 0</td>
</tr>
<tr>
<td>( s^2 )</td>
<td>( \frac{2 \times 6 - (1 \times 8)}{2} = +2 )</td>
<td>( \frac{2 \times 50 - (1 \times 0)}{2} = +50 )</td>
<td></td>
</tr>
<tr>
<td>( s^1 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s^0 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Compute $s^1$ row

Always use coefficients from 1st column

<table>
<thead>
<tr>
<th></th>
<th>$s^4$</th>
<th>$s^3$</th>
<th>$s^2$</th>
<th>$s^1$</th>
<th>$s^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>6</td>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^4$</td>
<td></td>
<td>2</td>
<td>8</td>
<td>(implicit) 0</td>
<td></td>
</tr>
<tr>
<td>$s^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Compute $s^0$ row

Always use coefficients from 1st column

<table>
<thead>
<tr>
<th></th>
<th>$s^4$</th>
<th>$s^3$</th>
<th>$s^2$</th>
<th>$s^1$</th>
<th>$s^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>6</td>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^4$</td>
<td></td>
<td>2</td>
<td>8</td>
<td>(implicit) 0</td>
<td></td>
</tr>
<tr>
<td>$s^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Final Step

<table>
<thead>
<tr>
<th>s^4</th>
<th>+1</th>
<th>6</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^3</td>
<td>+2</td>
<td>8</td>
<td>(implicit) 0</td>
</tr>
<tr>
<td>s^2</td>
<td>(\frac{(2 \cdot 6) - (1 \cdot 8)}{2} = +2)</td>
<td>(\frac{(2 \cdot 50) - (1 \cdot 0)}{2} = +50)</td>
<td></td>
</tr>
<tr>
<td>s^1</td>
<td>(\frac{(2 \cdot 8) - (2 \cdot 50)}{2} = -42)</td>
<td>(\frac{(2 \cdot 0) - (2 \cdot 0)}{2} = +0)</td>
<td></td>
</tr>
<tr>
<td>s^0</td>
<td>(\frac{(-42 \cdot 50) - (2 \cdot 0)}{-42} = +50)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Two sign changes in 1st column

Two roots with non-negative real parts

### Example Problem #2

\[s^4 + 10s^3 + 35s^2 + 50s + 24 = 0\]

<table>
<thead>
<tr>
<th>s^4</th>
<th>1</th>
<th>35</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^3</td>
<td>10</td>
<td>50</td>
<td>(implicit) 0</td>
</tr>
<tr>
<td>s^2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>s^1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>s^0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Example Problem #2

\[ s^4 + 10s^3 + 35s^2 + 50s + 24 = 0 \]

<table>
<thead>
<tr>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
<th>( s^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35</td>
<td>24</td>
<td>(implicit) 0</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>50</td>
<td>(10<em>35)-(1</em>50) = +30</td>
<td>(10<em>24)-(1</em>0) = +24</td>
<td></td>
</tr>
<tr>
<td>( \frac{(30<em>50)-(10</em>24)}{30} ) = +42</td>
<td>( \frac{(30<em>0)-(10</em>0)}{30} ) = +0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{(42<em>24)-(30</em>0)}{42} ) = +24</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

No sign changes in 1st column
No roots with non-negative real parts

### Example Problem #3

\[ s^5 + 11s^4 + 47s^3 + 97s^2 + 96s + 36 = 0 \]

<table>
<thead>
<tr>
<th>( s^5 )</th>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
<th>( s^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

No sign changes in 1st column
No roots with non-negative real parts
### Example Problem #3

\[ s^5 + 11s^4 + 47s^3 + 97s^2 + 96s + 36 = 0 \]

| \( s^5 \) | \( \frac{11 \times 47 - 1 \times 97}{11} = +38.182 \) | \( \frac{11 \times 96 - 1 \times 36}{11} = +92.727 \) |
| \( s^4 \) | 11 | 97 |
| \( s^3 \) | 47 | 96 |

- No sign changes in 1st column
- No roots with non-negative real parts

### Example Problem #4

\[ s^5 + 9s^4 + 27s^3 + 33s^2 + 26s + 24 = 0 \]

| \( s^5 \) | \( \frac{9 \times 27 - 1 \times 33}{9} \) | \( \frac{9 \times 26 - 1 \times 24}{9} \) |
| \( s^4 \) | 9 | 33 |
| \( s^3 \) | 27 | 26 |

- No root changes in 1st column
- No roots with non-negative real parts
### Example Problem #4

\[ s^5 + 9s^4 + 27s^3 + 33s^2 + 26s + 24 = 0 \]

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^5 )</td>
<td>1</td>
<td>27</td>
<td>26</td>
</tr>
<tr>
<td>( s^4 )</td>
<td>9</td>
<td>33</td>
<td>24</td>
</tr>
<tr>
<td>( s^3 )</td>
<td>+23.333</td>
<td>+23.333</td>
<td></td>
</tr>
<tr>
<td>( s^2 )</td>
<td>+24</td>
<td>+24</td>
<td></td>
</tr>
<tr>
<td>( s^1 )</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( s^0 )</td>
<td>???</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What do we do here?