2nd Order System Response

- **Typical 2nd order systems**
  - 2 energy storing elements
    - must store different types of energy!
  - 1 element which dissipates (or removes) energy
- **a spring - mass - damper combination**
- **a resistor - inductor - capacitor (RLC) combination**
Spring - Mass - Damper

State Variable Equations:
\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= \frac{1}{M} (-Kx - Bv + f(t))
\end{align*}
\]

Find the transfer function,
\[
\frac{X(s)}{F(s)} = \frac{1}{s^2 + \frac{B}{M} s + \frac{K}{M}}
\]

Simulink Simulation

\[
f(t) = 493 \, \text{N}, \, M = 2.5 \, \text{kg}, \, B = 16 \, \text{N/m/s}, \, K = 2465 \, \text{N/m}
\]
Simulation Set #1

Use Simulink to plot the position $X$ with these parameters ($f(t) = 493 \text{ N}$):

<table>
<thead>
<tr>
<th>M (kg)</th>
<th>K (N/m)</th>
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<tbody>
<tr>
<td>2.5</td>
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<td>2465</td>
<td>24</td>
</tr>
<tr>
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<td>32</td>
</tr>
<tr>
<td>2.5</td>
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<td>40</td>
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Questions about Set #1

• What parameter were we changing?
• What effects did this change have on the response of the system?
Generic 2nd Order System

\[ \frac{X(s)}{F(s)} = \frac{1}{M} \frac{1}{s^2 + \frac{B}{M}s + \frac{K}{M}} \rightarrow \frac{a_3}{s^2 + a_1s + a_2} \]

Find roots by quadratic formula

2nd Order Systems

- **Three possible solutions to quadratic equation**

  - 2 distinct, real roots \[ a_1^2 - 4a_2 > 0 \]
  - 2 repeated, real roots \[ a_1^2 - 4a_2 = 0 \]
  - 2 complex conjugate roots \[ a_1^2 - 4a_2 < 0 \]
2 complex conjugate roots

Roots of characteristic equation plotted on complex plane

\[ r_1, r_2 = \frac{-a_1 \pm j\sqrt{4a_2 - a_1^2}}{2} \]

\[ \times r_1 \]

- Real

\[ -\frac{a_1}{2} \]

\[ \times r_2 \]

+ Real

- Imaginary

+ Imaginary

2nd Order System - Complex Roots

\[ s^2 + a_1s + a_2 = 0 \]  Characteristic equation

\[ s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \]  Alternate form

\[ \zeta = \text{damping ratio} \quad \omega_n = \text{natural frequency} \]

\[ s_1, s_2 = -\zeta\omega_n \pm j\omega_n \sqrt{1-\zeta^2} \quad \zeta < 1 \text{ (for complex roots)} \]
2 complex conjugate roots

\[ r_1, r_2 = -\zeta \omega_n \pm j\omega_n \sqrt{1-\zeta^2} \]

\[ \times r_1 \quad \times r_2 \]

- Real

\[ -\zeta \omega_n \quad +\zeta \omega_n \]

+ Imaginary

\[ + j\omega_n \sqrt{1-\zeta^2} \quad - j\omega_n \sqrt{1-\zeta^2} \]

Complex Roots

\[ r_1 \]

\[ \omega_n \sqrt{1-\zeta^2} \]

\[ \phi = \cos^{-1} \zeta \]

- Imaginary

\[ - j\omega_n \sqrt{1-\zeta^2} \]

- Real

\[ -\zeta \omega_n \quad \zeta \omega_n \]

+ Real
Simulation Set #1

Plot the root locations for each of these sets of parameters

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Other Questions about Set #1

- What parameter were we changing?
- What effects did this change have on the response of the system?
- Do the 4 root locations have anything in common?
Simulation Set #2

Use Simulink to plot the position $X$ with these parameters ($f(t) = 493$ N):

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<td>16</td>
</tr>
<tr>
<td>3.5</td>
<td>2465</td>
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</tr>
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Questions about Set #2

- What parameter were we changing?
- What effects did this change have on the response of the system?
Simulation Set #2

Plot the root locations for each of these sets of parameters

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Other Questions about Set #2

- What parameter were we changing?
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Simulation Set #3

Use Simulink to plot the position $X$ with these parameters ($f(t) = 493$ N):

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2nd Order Systems

- **Three possible solutions to quadratic equation**
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  - 2 repeated, real roots $\Rightarrow a_1^2 - 4a_2 = 0$
  - 2 complex conjugate roots $\Rightarrow a_1^2 - 4a_2 < 0$
2 distinct, real roots

Roots of characteristic equation plotted on complex plane

\[ r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \]

2 repeated, real roots

Roots of characteristic equation plotted on complex plane

\[ r_1, r_2 = \frac{-a_1 \pm 0}{2} \]
Simulation Set #3

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