"Lumped Capacitance" method

> Valid when $B_\text{c} \ll 1$

(Say $B_\text{c} < 0.1$)

$$B_\text{c} = \frac{\text{internal } R}{\text{external } R} = \frac{L_\text{c}}{R}$$

(for convection)

$$L_\text{c} = \frac{\text{Volume}}{\text{Area for heat transfer}}$$

Example: Prob 5-10

- $D = 0.10 \text{ m}$
- Carbon-Steel
- $A_{ST} = 1010$
- $T_\infty = 1200 \text{ K}$
- $h = 100 \text{ W/m}^2\text{K}$
- $T_\text{c} = 300 \text{ K}$
- $\Delta t = ?$ when $T_{cL} = 800 \text{ K}$

Check to see if $L_\text{c}$ is okay:

if $B_\text{c} = \frac{hL_\text{c}}{T_\text{c}} \ll 1$ then okay

$$L_\text{c} = \frac{\text{Vol}}{A_{sur}} = \frac{\pi D^2/4 \times L_{rod}}{\left(\pi DL_{rod} + 2\left(\frac{\pi D^2}{4}\right)\right)}$$

"Sides" "ends"
So \[ L_c = \frac{D/4}{1 + \frac{D}{2L_{rod}}} \]

For \( D/L_{rod} \ll 1 \):

\[ L_c = \frac{D}{4} \]

Then:

\[ B_i = \frac{h(D/4)}{k} = \frac{100 \text{ W/m}^2 \text{K} \times (0.04)}{52 \text{ W/m}^2 \text{K}} \]

(For Table A1, use value at avg T = 550K)

\[ B_i = 0.048 \ll 1 \] so \( L_c \) is ok!!

\( L_c \rightarrow \theta(t) = \theta_0 e^{-t/\tau} \) for steel

\[ \tau = \text{time constant} = \frac{(90) + (180)}{90} \times \frac{100 \text{ W/m}^2 \text{K}}{100 \text{ W/m}^2 \text{K}} \times 0.025 \text{ m} \]

\[ \tau = 1095.5 \text{ s} = 0.81 \text{ min} \]

\[ t_{reqd} = \ln\left(\frac{\theta_{reqd}}{\theta_0}\right) \times (-\tau) = \ln\left(\frac{800 - 1200}{300 - 1200}\right) \times (-1095.5 \text{ s}) \]

\[ t_{reqd} = 888 \text{ s} = 0.25 \text{ h} \]

What if \( B_i \approx 1 \) or greater?
Then, must take account of internal $T$ variation.

(See Fig. 5.4 on page 260)

Go back to our heat conduction equation. Consider Cartesian geometry, 1-D, transient, constant properties, no internal generation.

$$\frac{\partial^2 T}{\partial x^2} = \frac{\rho c}{k} \frac{\partial T}{\partial t} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

Note $\frac{k}{\rho c} = \alpha$, thermal diffusivity $\frac{m^2}{s}$ or $\frac{ft^2}{s}$

Consider slab uniformly heated by convection on both sides

Note: symmetric heating slab 2L

Same as one side heating slab L, adiabatic on other side

Let $\Theta(x,t) = T(x,t) - T_{\infty}$

$$\frac{\partial^2 \Theta}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \Theta}{\partial t}$$

Note this is a partial differential equation.
With B.C.s
\[ \frac{\partial \theta}{\partial x} \bigg|_{x=0} = 0 \]
\[ -k \frac{\partial \theta}{\partial x} \bigg|_{x=L} = h \theta \bigg|_{x=L} \]
\[ t > 0 \]

And
\[ \theta(x, 0) = \theta_0 = T(x, 0) - T_\infty \]

To solve this P.D.E., assume \( \theta(x, t) \) has this form
\[ \theta(x, t) = X(x) T(t) \]

Then
\[ \frac{\partial^2 \theta}{\partial x^2} = \frac{d^2 X}{dx^2} T(t) = X'' T \]

And
\[ \frac{\partial \theta}{\partial t} = X(x) \frac{dT}{dt} = X T' \]

Substituting into P.D.E. gets
\[ \frac{X}{X} = \frac{1}{\alpha} \frac{1}{T} = \frac{1}{T} T' = -\lambda_n \quad \alpha \text{ a constant} \]

Eigenvalue (determined by the eigenvalue condition for the problem)

For the slab
\[ \theta(x,t) = \sum_{n=1}^{\infty} C_n \cos \left( \frac{\lambda_n x}{L} \right) \exp \left( -\frac{\lambda_n^2 dt}{L^2} \right) \]

Where
\[ C_n = \frac{4 \sin \lambda_n}{2 \lambda_n + \sin(2 \lambda_n)} \]
and \( \lambda_n \)'s satisfy
\[
\lambda_n + \tan(\lambda_n L) = \frac{\hbar L}{k}
\]
eigenvalue condition

\[
\tan(\lambda_n L) = \frac{\hbar L}{\lambda_n k}
\]

\[
\frac{B \ell}{\lambda_n}
\]